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SEQUENTIAL PROCEDURES FOR ESTIMATING THE MEAN OF AN INVERSE GAUSSIAN DISTRIBUTION

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SUMMARY

Sequential procedures are proposed for the point estimation of the mean of an inverse Gaussian distribution under quadratic loss structures. Two cases are discussed separately, viz. (i) when the stopping rule depends on the estimator of only one nuisance parameter, and (ii) when the stopping rule depends on the estimators of two nuisance parameters. In both the cases, the proposed procedures are shown to be asymptotically 'risk-efficient' in the sense of Starr [13]. Moreover, a sequential procedure is also developed to construct fixed-width confidence interval for the mean. The procedure is shown to be 'asymptotically efficient' and 'asymptotically consistent' in view of Chow and Robbins [6].

Keywords : Point estimation, Quadratic loss, Stopping times, Uniform integrability, Risk—efficiency, Fixed-width interval, Asymptotic efficiency.

Introduction

Suppose that $X_1, X_2, \ldots$ is a sequence of i.i.d. random observations from an inverse Gaussian population

$$f(x; \mu, \lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left[-\frac{\lambda}{2\mu^2} \cdot \frac{(x-\mu)^2}{x}\right] \; (x > 0) \qquad (1.1)$$

where μ (> 0) and λ (> 0) are population parameters. Recently, Chaturvedi [2] developed a sequential procedure for estimating μ with prescribed 'proportional closeness'. The proposed procedure was shown to be

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'asymptotically efficient' and 'asymptotically consistent' in the sense of Chow and Robbins [5]. In a later communication, Chaturvedi [3] obtained second-order approximations for the sequential procedure.

In the present note, a sequential procedure has been developed for the point estimation of μ under quadratic loss function plus linear cost of sampling. Two cases will be dealt separately, (i) when λ is known and (ii) when λ is not known and which will yield two differential stopping rules for the same estimation problem. Asymptotic behaviour of the 'risk-efficiency' (see Starr [13] for definition) of the proposed procedures is studied. A sequential procedure has been obtained to construct fixed-width confidence interval for μ without any 'proportional closeness' concept. It is remarkable that the estimation procedures adopted here are quite different from that of Chaturvedi ([5], [6]) in the sense that here the stopping rule and estimation rule are highly dependent on each other and, as sucn, separate analysis is required to prove various results.

2. Point Estimation of μ

Having recorded a random sample $X_1, \ldots, X_n$ of size n, let the loss incurred in estimating μ by $\bar{X}_n = n^{-1} \sum_{i=1}^{n} X_i$ be squared-error plus linear cost of sampling, i.e.,

$$L_n = A\,(\bar{X}_n - \mu)^2 + n \tag{2.1}$$

where A (> 0) is the known weight and without any loss of generality, it has been assumed that cost of sampling per unit observation is unity. The risk corresponding to the loss (2.1) is

$$v_n(A) = \frac{A\mu^3}{n\,\lambda} + n \tag{2.2}$$

When μ and λ both are known, the fixed sample size n_0, which minimizes $v_n(A)$, is given by

$$n_0 = (A/\lambda)^{1/2}\,\mu^{3/2} \tag{2.3}$$

and setting $n = n_0$ in (2.2), the corresponding minimum risk is

$$v_{n_0}(A) = 2\,(A/\lambda)^{1/2}\,\mu^{3/2} \tag{2.4}$$

However, if μ and/or λ is unknown, there does not exist any fixed sample size procedure which minimizes the risk. In these situations, purely sequential procedures are adopted.

2.1. *The Case when λ is Known*

Without loss of generality, let us take $\lambda = 1$. Motivated by (2.3), we adopt the following stopping rule.

The stopping time $N = N_A$ is the smallest positive integer n such that

$$N = N_A = \inf\,[n \geqslant 1 : n \geqslant A^{1/2}\, \bar{X}_n^{3/2}] \tag{2.5}$$

Following Starr [13], define the 'risk-efficiency' of the sequential procedure (2.5) by

$$\eta_A = \nu_N(A)/\nu_{n_0}(A) \tag{2.6}$$

where $\nu_{n_0}(A)$ can be obtained from (2.4), letting $\lambda = 1$ and $\nu_N(A)$ is the risk associated with the sequential procedure, i.e.

$$\nu_N(A) = AE(\bar{X}_N - \mu)^2 + E(N) \tag{2.7}$$

Setting $S_n = n\,\bar{X}_n$, the stopping rule (2.5) can be equivalently written as

$$N = N_A = \inf\,[n \geqslant 1 : S_n \leqslant A^{-1/3} n^{5/3}] \tag{2.8}$$

Now prove the following theorem which establishes the result that the sequential procedure (2.5) or (2.8) is asymptotically risk-efficient.

THEOREM 1 : *For the stopping rule defined at* (2.5) [or (2.8)] *and all* $\mu > 0$, $\lim_{A \to \infty} \eta_A = 1$.

PROOF : By Wald's lemma for cumulative sums (see, Theorem 2 of Chow, Robbins and Teicher [6]),

$$E\{(S_N - N\mu)^2\} = \mu^2 E(N)$$

Hence we obtain from (2.7),

$$\begin{aligned}\nu_N(A) &= E\{AN^{-2}(S_N - N\mu)^2\} + E(N)\\ &= E\{(S_N - N\mu)^2 (AN^{-2} - \mu^{-2})\} + 2E(N)\end{aligned} \tag{2.9}$$

Making substitutions from (2.4) and (2.9) in (2.6), one gets

$$\eta_A = \tfrac{1}{2} A^{-1/2} \mu^{-3/2} E\{(S_N - N\mu)^2 (AN^{-2} - \mu^{-2})\} + \mu^{-3/2} E(A^{-1/2} N) \tag{2.10}$$

It is easy to verify that $A^{-1/2} N \to \mu^{3/2}$ a.s. as $A \to \infty$. From Gut [9], it follows that

$$\{(A^{-1/2} N)^4 : A \geqslant 1\} \text{ is uniformly integrable} \tag{2.11}$$

Thus, from (2.11) and dominated convergence theorem, $\mu^{-3/2}. E(A^{-1/2} N) \to 1$ as $A \to \infty$ and hence, by (2.10), the theorem follows if it can be proved that

$$E\{A^{-1/2}(S_N - N\mu)^2 (AN^{-2} - \mu^{-3})\} = 0\,(1) \text{ as } A \to \infty \tag{2.12}$$

To this end, we have by Hölder's inequality,

$$A^{-1/2}\,\mathrm{E}\,|\,(S_N - N\mu)^2 (AN^{-2} - \mu^{-3})\,|$$

$$\leqslant E^{1/2}\,|\,A^{-1/4}(S^N - N\mu)\,|^4\, E^{1/2}\,|\,AN^{-2} - \mu^{-3}\,|^2 \tag{2.13}$$

From (2.11) and Lemma 5 of Chow and Yu [8]

$$\{\,|\,A^{-1/4}(S_N - N\mu)\,|^4\; A \geqslant 1\} \text{ is uniformly integrable} \tag{2.14}$$

It follows from the definition of N that $A^2 N^{-4} \leqslant (N^{-1} S_N)^{-6}$. Hence, using the dominated ergodic theorem of Marcinkiewicz and Zygmund (see, Chow and Teicher [7] p. 357],

$$\{(A^2 N^{-4}) : A \geqslant 1\} \text{ is uniformly integrable} \tag{2.15}$$

Utilizing the results $A^{1/2}\ N^{-1} \to \mu^{-3/2}$ a.s as $A \to \infty$, (2.14) and (2.15), we obtain from (2.13),

$$A^{-1/2}\,E\,|\,(S_N - N\mu)^2 (AN^{-2} - \mu^{-3})\,|$$

$$\leqslant 0\,(1).\ E^{1/2}\,|\,AN^{-2} - \mu^{-3}\,|^2$$

$$= 0\,(1) \text{ as } A \to \infty$$

This completes the proof of the theorem.

2.2 *The Case when λ is Unknown*

In the situation when λ is also unknown, the stopping rule (2.5) can't be used for the estimation of μ. We estimate λ^{-1} by $\lambda_n = (n-1)^{-1} \sum_{i=1}^{n} (X_i^{-1} - \bar{X}_n^{-1})$ (see, Johnson and Kotz [10], p. 137) and define the following stopping rule in conformity with (2.3) :

$$N^* = N^*_A = \inf [n \geqslant m : n \geqslant (A\lambda_n)^{1/2} \bar{X}_n^{3/2}]$$

$$= \inf [n > m : S_n \leqslant (A\lambda_n)^{-1/3}. n^{5/2}] \quad (2.16)$$

where m being the starting sample size and is such that, for $\delta > 0$, $\delta A^{1/4} \leqslant m = 0(A^{1/2})$ as $A \to \infty$.

The 'risk-efficiency' of the sequential procedure (2.16) is defined by

$$\eta^* = \nu_{N^*}(A)/\nu_{n_0}(A) \quad (2.17)$$

where $\nu_{n_0}(A)$ is same as that defined at (2.4) and $\nu_{N^*}(A)$ is the risk corresponding to the sequential procedure (2.16), i.e.

$$\nu_{N^*}(A) = AE(\bar{X}_{N^*} - \mu)^2 + E(N^*) \quad (2.18)$$

The main result of this sub-section is stated in the following theorem.

THEOREM 2 : *For all* $(\mu, \lambda) > 0, \lim_{A\to\infty} \eta^*_A = 1$

PROOF : Appllying Theorem 2 of Chow, Robbins and Teicher [6], it follows that

$$\nu_{N^*}(A) = E\{(S_{N^*} - N^*\mu)^2 (AN^{*^{-2}} - \lambda\mu^{-3})\} + 2E(N^*)$$

It is easy to see that $A^{-1/2} N^* \to \lambda^{-1/2} \mu^{3/2}$ a.s. as $A \to \infty$. It follows from Lemma 3 of Martinsek [12] that under the condition imposed on the starting sample size m at (2.16), for $P > 0$

$$\{(A^{-1/2} N^*)^{-p} : A \geqslant 1\} \text{ is uniformly integrable} \quad (2.19)$$

Applying the result (2,19), we obtain

$2E(N^*)/\nu_{n_0}(A) = (\lambda^{1/2}\mu^{-3/2}) \; E(A^{-1/2} N^*) \to 1$ as $A \to \infty$, and, it therefore remains to prove that

$$A^{-1/2} E\{(S_{N^*} - N^*\mu)^2 (AN^{*-2} - \lambda\mu^{-3})\} = 0(1) \text{ as } A \to \infty \quad (2.20)$$

To this end, by Holder's inequality,

$$E \mid A^{-1/2}\{(S_{N^*} - N^*\mu)^2 (AN^{*-2} - \lambda\mu^{-3})\} \mid$$

$$\leqslant E^{2/1} \mid A^{-1/4}(S_{N^*} - N^*\mu) \mid^4 . \; E^{1/2} \mid AN^{*-2} - \lambda\mu^{-3}/2 \quad (2.21)$$

From Lemma 5 of Chow and Yu [8]

$$\{ | A^{-1/4} (S_{N^*} - N^* \mu) |^4 : A \geqslant 1\} \text{ is uniformly integrable} \tag{2.22}$$

Since $A^{1/2} N^{*-1} \to \lambda^{1/2} \mu^{-3/2}$ a.s. as $A \to \infty$, utilizing (2.19) and (2.22), we obtain from (2,21),

$$E | A^{-1/2} \{(S_{N^*} - N^* \mu)^2 (AN^{*-2} - \lambda \mu^{-3})\} |$$

$$\leqslant 0 (1). E^{1/2} | AN^{*-2} - \lambda \mu^{-3} |^2$$

$$= 0 (1) \text{ as } A \to \infty$$

and (2.20) follows.

3. Fixed-Width Confidence Interval for μ

For specified d (> 0) and $\alpha \varepsilon (0,1)$, suppose one wishes to construct an interval I_n for μ such that the width of I_n is $2 d$ and $P (\mu \in I_n) \geqslant 1 - \alpha$. Let us define $I_n = (\bar{X}_n - d, \bar{X}_n + d)$. Using the fact that $\sqrt{n \lambda}$ $\mu^{-3/2} (\bar{X}_n - \mu) \alpha \overset{\alpha}{\to} N (0,1)$, we obtain

$$P (\mu \in I_n) = 2 \Phi (d \sqrt{n \lambda} \mu^{-3/2}) - 1 \tag{3.1}$$

where $\Phi (\cdot)$ stands for the c.d.f. of a $N (0, 1)$ r.v. Let 'a' be any constant such that

$$2 \Phi (a) - 1 = 1 - \alpha \tag{3.2}$$

It is obvious from (3.1) and (3.2) that for known μ and λ, in order to achieve $P (\mu \in I_n) \geqslant 1 - \alpha$, the fixed sample size required is the smallest positive integer $n \geqslant n^*$, where $n^* = a^2 \mu^3/d^2 \lambda$. But, in the absence of any knowledge about μ and λ, no fixed sample size procedure will achieve the requirements simultaneously for all values of μ and λ. In such a situation, adopt a sequeutial procedure, which could be described as follows.

Let us start with a sample of size m ($\geqslant 2$). Then, the stopping time $N = N (d)$ is the smallest positive integer $n > m$ for which

$$n > a^2 \bar{X}_n^3 \lambda_n / d^2 \tag{3.3}$$

When stop, construct $I_N = (\bar{X}_N - d, \bar{X}_N + d)$ for μ.

Now, prove the following theorem which establishes the results that the sequential procedure (3.3) is 'asymptotically efficient' and 'asymptotically consistent' in Chow-Robbins [5] sense.

THEOREM 3 : *N is a well-defined stopping time, non-decreasing as a function of d, and*

$$\lim_{d \to 0} N = \infty \text{ a.s.} \tag{3.4}$$

$$\lim_{d \to 0} (N/n^*) = 1 \text{ a.s.} \tag{3.5}$$

$$\lim_{d \to 0} E(N/n^*) = 1 \tag{3.6}$$

$$\lim_{d \to 0} P(\mu \in I_N) = 1 - \alpha \tag{3.7}$$

PROOF : Result (3.4) follows from the definition of N. Note the inequality

$$(a^2 \bar{X}_N^3 \lambda_N/d^2) \leqslant N \leqslant (a^2 \bar{X}_N^3 \lambda_N/d^2) + (m-1) \tag{3.8}$$

or,

$$(\bar{X}_N/\mu)^3 \cdot \lambda\lambda_N \leqslant (N/n^*) \leqslant (\bar{X}_N/\mu)^3 \cdot \lambda\lambda_N + (m-1)/n^*$$

which, on using (3.4) and the facts that $\lim_{N \to \infty} \bar{X}_N = \mu$ a.s.,

$\lim_{N \to \infty} \lambda_N = \lambda^{-1}$ a.s, and $\lim_{d \to 0} n^* = \infty$ a.s., leads us to (3.5).

Since, for all N, $\bar{X}_N$ and λ_N are stochastically independent (see, Chaturvedi [2]), using the fact that

$(N-1)\lambda\lambda_N = \sum_{j=1}^{N-1} Y_j$, with $Y_j \sim \chi^2(1)$. we obtain from (3.8)

$$E(N) \leqslant a^2 E\{N^{-3} (\sum_{i=1}^{N} X_i)^3\} E\{(N-1)^{-1} \sum_{j=1}^{N-1} Y_j\} + (m-1) \tag{3.9}$$

Since $E\{N^{-3}(\sum_{i=1}^{N} X_i)^3\} \leqslant E\{\sup_{N \geqslant 1} N^{-3} |\sum_{i=1}^{N} X_i|^3\}$ and $E|X_i|^3 < \infty$ for all $(\mu, \lambda) \leqslant \infty$ (see, Johnson and Kotz [10], p. 137), it follows from

Wiener ergodic theorem (see, Khan [11]) that $E\{\sup_{N\geqslant i} N^{-3}(\sum_{i=1}^{N} X_i)^3\} < \infty$. By similar arguments, we can show that $E\{\sup_{N\geqslant 2}(N-1)^{-1}(\sum_{j=1}^{N-1} Y_j\} < \infty$. Hence, it is concluded from (3.9) that N is uniformly integrable and (3.6) follows from (3.5) and dominated convergence theorem.

In the notations of Theorem 1 of Chaturvedi [4] (due to Anscombe [1]), let $Z_n = \overline{X}_n$, $\theta = \mu$, $\tau_n = \mu^{3/2}/\sqrt{n\lambda}$ $\hat{\tau}_n = \overline{X}_n^{3/2}\lambda_n^{1/2}/\sqrt{n}$, $t = \sqrt{n}$, $a(t) = d^2\sqrt{n\lambda}/a^2\mu^{3/2}$, $n(t) = n^*$ and $N(t) =$ the stopping rule, it follows that all the conditions of the theorem are satisfied and hence, $\sqrt{N\lambda}(X_N - \mu)/\mu^{3/2} \xrightarrow{n} N(0,1)$ as $d \to 0$. Thus, using (3.5), we obtain

$$\lim_{d\to 0} P(\mu \in I_N) = \lim_{d\to 0} P\{|N(0,1)| \leqslant d\sqrt{N\lambda}/\mu^{3/2}\}$$

$$= P\{|N(0,1)| \leqslant d\sqrt{n^*\lambda}/\mu^{3/2}\}$$

$$= P\{|N(0,1)| \leqslant a\}$$

and (3.7) holds.

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